

MULTIPHASE SOLUTIONS TO THE VECTOR ALLEN-CAHN EQUATION

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1. MOTIVATION

It is common knowledge that the same substance can exist in different forms or *phases* and that, depending on temperature and pressure, two or more phases of the same substance can coexist at thermodynamical equilibrium.

In the case of alloys or mineral compounds one can observe regions occupied by two different crystalline phases separated by an *interface* or there may be singular points, i.e. *junctions*, where three or more equally preferred phases meet.

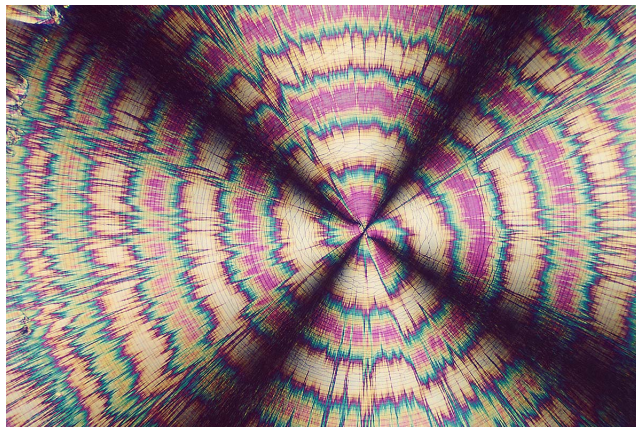


FIGURE 1. Hippuric acid (David Malin): a junction point. The junction point lays at the intersection of two reflection lines that correspond to the interfaces between four regions.

Another interesting structure is observed when polyhedral regions with different phases tile space with an almost regular pattern. Examples of such situations observed experimentally are presented in Figures 2 and 3.

The aim of our paper [3] is to discuss a simple mathematical model that can describe an amazing variety of multiphase states that include multiple junctions and tilings. A nonnegative function $W : \mathbb{R}^m \rightarrow \mathbb{R}^+$, $m > 1$, with a finite number of zeros is used to model the bulk free energy

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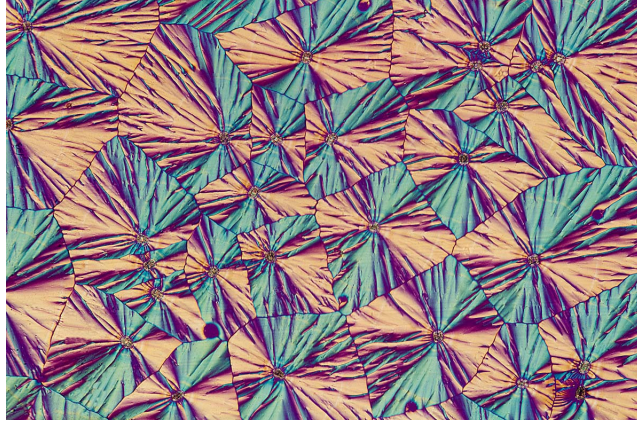


FIGURE 2. Tiling, Vitamin C crystals (David Malin). Tiling of the plane with structured multiphase cells. Note the junction point in each cell.

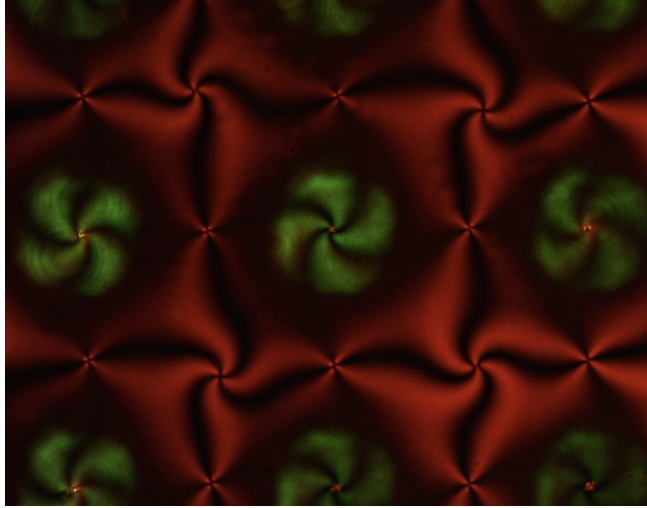


FIGURE 3. Square lattice structure of a liquid crystal illuminated by periodic laser lights.

density of a material that, at thermodynamical equilibrium, can exist in several equally preferred phases corresponding to the zeros $a_1, \dots, a_N \in \mathbb{R}^m$ of W . In some cases the vector u describes the mass fraction of each element but the number of minimal energy phases, N , may or may not equal m . Sometimes the different phases are different orientations of a crystal or different crystalline structures of the same elements at roughly the same concentrations.

Under the simplifying assumption that the interfacial energy density can be modeled by the isotropic quantity $\frac{\epsilon^2}{2} |\nabla u|^2$, where $0 < \epsilon \ll 1$ is a parameter that measures the importance of interfacial versus bulk energy, the free energy of the material contained in a bounded region $\Omega \subset \mathbb{R}^n$,

$n \geq 1$, can be modeled by the Allen-Cahn functional:

$$(1.1) \quad J_\Omega(u) = \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \right) dx,$$

where the map $\Omega \ni x \rightarrow u(x) \in \mathbb{R}^m$ describes the distribution of an *order parameter* $u(x) \in \mathbb{R}^m$ in Ω : $u(x) \approx a_j$ means that at $x \in \Omega$ the material is approximately in phase a_j and $u(x) \in \mathbb{R}^m \setminus \{a_1, \dots, a_N\}$ means that at x it is in some mixed or disordered state. To model the tendency of the system to evolve toward states of minimal energy we can consider the gradient system associated with J_Ω . This depends on the choice of the Hilbert space H that gives the metric with respect to which we compute the gradient of J_Ω . For $H = L^2(\Omega; \mathbb{R}^m)$ and writing $W_u(u)$ for $(\frac{\partial W(u)}{\partial u_1}, \dots, \frac{\partial W(u)}{\partial u_m})^\top$ we obtain the parabolic Allen-Cahn system of equations

$$(1.2) \quad \begin{cases} u_t = \epsilon^2 \Delta u - W_u(u), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \\ u(0, x) = u_0. \end{cases}$$

This system of nonlinear partial differential equations is probably the simplest model for phase separation phenomena. Given an initial state u_0 , for small $\epsilon > 0$, we can distinguish different behaviors in the dynamics of (1.2). In an initial time interval of order $O(1)$ the evolution is essentially dictated by the ODE $u_t = -W_u(u)$ that, depending on the structure of u_0 , evolves the solution $u^\epsilon(t, x, u_0)$ towards the set of minima of W , and as a result Ω is partitioned into subregions where u^ϵ is approximately constant and equal to one of the a_j . That is, the space is divided in subregions occupied by different coexisting phases of the material. These subregions are separated by an interface of thickness $O(\epsilon)$ across which u^ϵ makes a transition from a neighborhood of one of the minima of W to a neighborhood of another. Following this first period, the so called *separation* stage, u^ϵ develops high gradients across the interfaces and the two terms on the right hand side of (1.2) become comparable throughout Ω , and a second, slower, evolution of the material structure begins during which u^ϵ keeps its partitioned structure while the *phase boundaries* evolve with speed of order $O(\epsilon^2)$. This evolution stage often leads to a reduction of the measure and the disappearance of one or more connected components of the subregion corresponding to one of the phases. This phenomenon is called *coarsening* and involves changes in the topological structure of the interface. In any case, the asymptotic fate of the evolution is a solution $u : \Omega \rightarrow \mathbb{R}^m$ of the stationary vector Allen-Cahn equation

$$\begin{aligned} \epsilon^2 \Delta u &= W_u(u) \quad \text{for } x \in \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which is the Euler-Lagrange equation of the functional (1.1).

We are interested in the local structure of the phase-junction points where three or more subregions corresponding to different phases meet. Therefore, we focus on one of these points that we assume to coincide with the origin $0 \in \mathbb{R}^n$ and *dilate* the space around 0 via the transformation $x \rightarrow \epsilon x$. This, in the limit $\epsilon \rightarrow 0$, leads to the study of entire solutions $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of the rescaled equation

$$(1.3) \quad \Delta u = W_u(u), \quad x \in \mathbb{R}^n,$$

which is formally the Euler-Lagrange equation of the functional

$$(1.4) \quad J(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx.$$

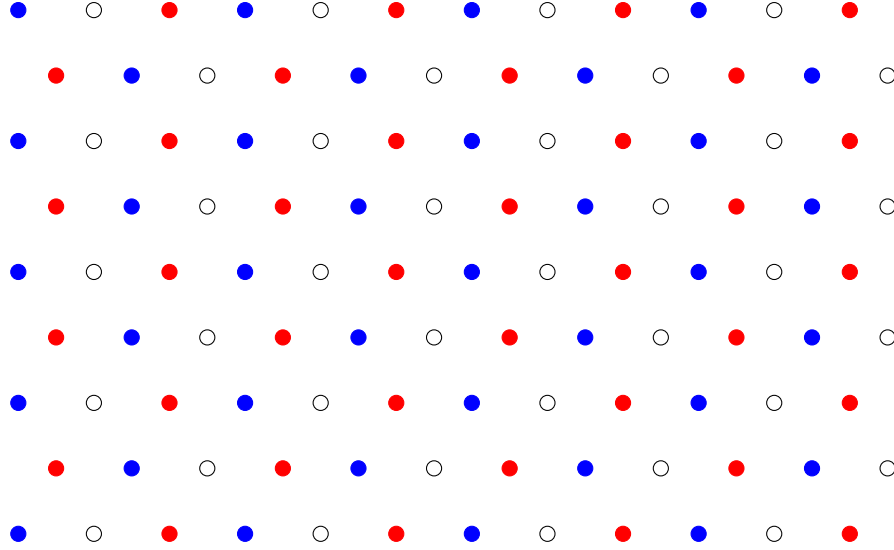


FIGURE 4. The symmetry of the underlying microscopic lattice which is consistent with W having the symmetry of a triangle with minima at the red, white, and blue states.

Entire solutions of (1.3) are also relevant to model the tiling of space observed in physical experiments.

The study of general entire solutions of (1.3) is very hard. The main difficulty is that, if W has two or more zeros, J_Ω is not convex and there is no a priori method for characterizing the regions where $u(x)$ is near to one or another of the a_1, \dots, a_N . We restrict to symmetric potentials W and to symmetric solutions. The restriction to the symmetric setting has both physical and mathematical motivations. From the physical point of view we observe that symmetry is ubiquitous in nature. Symmetric structures are observed at the junction of three or four coexisting phases in physical space. Similar structures appear at the singularities of soap films and compounds of soap bubbles. We also note that in the modeling of crystalline phase transitions of alloys it is quite natural to assume that W enjoys some kind of symmetry that reflects the symmetry of the underlying microscopic crystal lattice of the alloy. In such situations W can be assumed to depend on a vector order parameter $u \in \mathbb{R}^m$ which describes the fraction of the components of the alloy in each of m sublattices of the microscopic crystal lattice, and $u = a_j$, $j = 1, \dots, N$ corresponds to a pure phase [4]. To give the idea consider the planar lattice shown in Fig. 4. Assume that the lattice is composed of three sublattices: white, blue and red. In this situation the potential W is a function of the atomic fractions u_1, u_2 and u_3 of the white, blue and red sublattices. If the atomic interaction is the same independently of the color of the atoms, then W satisfies

$$\begin{aligned} W(u_1, u_2, u_3) &= W(u_2, u_3, u_1) = W(u_3, u_1, u_2), \\ W(u_1, u_2, u_3) &= W(u_1, u_3, u_2). \end{aligned}$$

Therefore W is invariant under the symmetry group of the equilateral triangle with vertices in $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ that may coincide with the zeros a_1, a_2, a_3 of W .

2. THE MAIN THEOREMS

In [3] we present a systematic study of symmetric bounded entire solution of (1.3). We assume that W is invariant under a finite reflection group Γ acting on \mathbb{R}^m and that there is a reflection group G acting on the domain space \mathbb{R}^n . Since we intend to include also periodic patterns we consider both the cases where G is a finite or an infinite (discrete) reflection group.

We assume that G and Γ are related by a homomorphism $f : G \rightarrow \Gamma$ and define a map $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be f -equivariant if

$$(2.1) \quad u(gx) = f(g)u(x), \quad \text{for } g \in G, x \in \mathbb{R}^n.$$

This notion of equivariance with respect to a homomorphism describes all symmetric patterns. Let us give some basic examples when $n = m = 1$.

- (1) Take $G = \Gamma = \{I, \sigma\}$, with $I : \mathbb{R} \rightarrow \mathbb{R}$ the identity map, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ the antipodal map $\sigma(x) = -x$, and let $f : G \rightarrow G$ be the identity. Then (2.2) takes the form

$$u(\sigma(x)) = \sigma(u(x)) \quad \Leftrightarrow \quad u(-x) = -u(x),$$

and a map $u : \mathbb{R} \rightarrow \mathbb{R}$ is f -equivariant if and only if it is odd.

- (2) With $G = \Gamma = \{I, \sigma\}$ as in (i), let f be the trivial homomorphism that maps σ into I . In this case (2.2) reads

$$u(\sigma(x)) = u(x) \quad \Leftrightarrow \quad u(-x) = u(x)$$

and a map u is f -equivariant if and only if it is even.

- (3) Next, we consider the discrete reflection group G of \mathbb{R} generated by $s_0 = \sigma$ and by the reflection $s_1(x) = 2 - x$. The elements of G are the reflections $s_k(x) = 2k - x$ and the translations $t_k(x) = x + 2k$ ($k \in \mathbb{Z}$). We take $\Gamma = \{I, \sigma\}$ as before, and define f such that $f(s_k) = \sigma = s_0$, $f(t_k) = I$. In this case (2.2) takes the form

$$\begin{aligned} u(s_k(x)) &= s_0(u(x)) \quad \Leftrightarrow \quad u(2k - x) = -u(x), \quad k \in \mathbb{Z}, \\ u(t_k(x)) &= I(u(x)) \quad \Leftrightarrow \quad u(2k + x) = u(x), \quad k \in \mathbb{Z}. \end{aligned}$$

The function $u(x) = \sin(\pi x)$ which satisfies these conditions, is an example of f -equivariant map, and the same is true for any odd 2-periodic map.

Similarly, when $n = m = 2$ the finite reflection groups of the plane are the dihedral groups D_k ($k \geq 1$), which are the symmetry groups of regular polygons. A basic example of discrete reflection group of the plane is the group G generated by the reflections along the lines $x_2 = k$ and $x_1 = l$ ($k, l \in \mathbb{Z}$) cf. Figure 5 left. If we take $G = \Gamma = D_2$ (cf. Figure 5 right), and f to be the identity, then a f -equivariant map $u = (u_1, u_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies the conditions:

$$\begin{aligned} u_1(-x_1, x_2) &= -u_1(x_1, x_2), \quad u_2(-x_1, x_2) = u_2(x_1, x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2, \\ u_1(x_1, -x_2) &= u_1(x_1, x_2), \quad u_2(x_1, -x_2) = -u_2(x_1, x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

In particular, a point belonging to the x_i ($i = 1, 2$) coordinate axis is mapped on the u_i coordinate axis, and $u(0, 0) = (0, 0)$.

The reflection hyperplanes of a finite or discrete reflection group partition the space into identical cells that are called *fundamental domains*. For instance, the square $F = (0, 1) \times (0, 1)$ is a fundamental domain of the discrete reflection group depicted in Figure 5 left, while a sector of angle π/k is a fundamental domain of the dihedral group D_k (cf. Figure 5 right when $k = 2$). In the one dimensional example (1) above, the half-line $F = (0, \infty)$ is a fundamental domain of

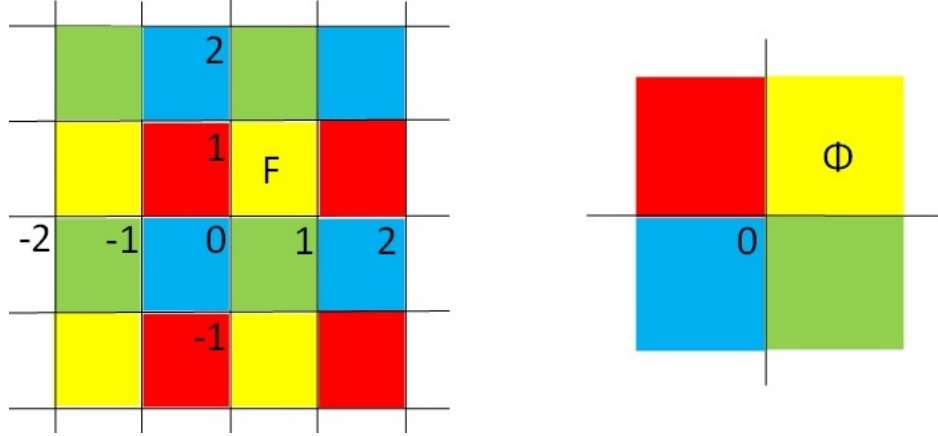


FIGURE 5. On the left: a square lattice generated by the reflections along the lines $x_2 = k$ and $x_1 = l$ ($k, l \in \mathbb{Z}$) parallel to the coordinate axes. On the right: the dihedral group D_2 generated by the two reflections along the coordinate axes.

$G = \Gamma = \{I, \sigma\}$. Finally, in example (3) above, the interval $F = (0, 1)$ is a fundamental domain of the corresponding discrete reflection group G .

We characterize the homomorphisms which allow for the existence of f -equivariant maps that send a *fundamental domain* F for the action of G on \mathbb{R}^n into a *fundamental domain* Φ for the action of Γ on \mathbb{R}^m :

$$(2.2) \quad u(\overline{F}) \subset \overline{\Phi}.$$

In the case of the one dimensional example (1) above, condition (2.2) means that $u(x) \geq 0$, $\forall x \in \overline{F} = [0, \infty)$. Similarly, in the case of example (3) above, condition (2.2) means that $u(x) \geq 0$, $\forall x \in \overline{F} = [0, 1]$. By analogy with the scalar case $m = 1$, and for the coherence of the terminology, we refer to these homomorphisms and to the maps that satisfy (2.2) as *positive*.

If, as in [3], we assume that W is invariant under Γ and that W has a unique zero, say a_1 , in $\overline{\Phi}$, from (2.2) it follows that, for $x \in \overline{F}$, $u(x)$ remains at a distance from all the zeros of W different from a_1 . This is of central importance in the proofs of Theorem 2.1 and Theorem 2.2 below. Not all homomorphisms allow for the existence of nontrivial f -equivariant maps that satisfy (2.2). To see that positive homomorphisms need to satisfy a particular condition observe that, if u is an f -equivariant satisfying (2.2), and $x \in \overline{F}$ belongs to P , one of the hyperplanes that bound F , we have

$$u(x) = u(s(x)) = f(s)u(x) \in \overline{\Phi},$$

where $s \in G$ is the reflection in P . It follows that $u(x) \in \overline{\Phi}$ is a fixed point of $f(s)$ and therefore, since $u(x)$ in the interior of Φ implies $\gamma u(x) \neq u(x)$ for any $\gamma \in \Gamma \setminus \{I_m\}$, we necessarily have that $u(x)$ belongs to one of the hyperplanes that bound $\overline{\Phi}$. The conclusion is that $f(s)$ is either one of the reflections associated to the hyperplanes that bound $\overline{\Phi}$ or the product of some of these reflections. Therefore we define

Definition 1. Let F be a fundamental domain of G , bounded by the hyperplanes P_1, \dots, P_l , corresponding to the reflections s_1, \dots, s_l . We say that a homomorphism $f : G \rightarrow \Gamma$ is *positive* if

there exists a fundamental domain Φ of Γ , bounded by the hyperplanes Π_1, \dots, Π_k , such that for every $i = 1, \dots, l$, there is $1 \leq k_i \leq k$ and $\tilde{\Pi}_1, \dots, \tilde{\Pi}_{k_i} \in \{\Pi_1, \dots, \Pi_k\}$ such that

$$(2.3) \quad \ker(f(s_i) - I_m) = \cap_{j=1}^{k_i} \tilde{\Pi}_j.$$

That is, the set of points fixed by the orthogonal map $f(s_i)$ is one of the hyperplanes Π_j , or the intersection of several of them.

Positive homomorphisms have certain mapping properties that relate the reflections associated to the walls of a fundamental domain F to the reflections associated to the walls of a corresponding region Φ . These properties are instrumental to show that minimizing in the class of f -equivariant maps that satisfy (2.2) does not affect the Euler-Lagrange equation and renders a smooth solution of (1.3). The proof of this fact is based on a quite sophisticated use of the maximum principle for parabolic equations. We prove that, provided f is a positive homomorphism, the L^2 gradient flow associated to the functional (1.4) preserves the positivity condition (2.2). By a careful choice of certain scalar projections of the vector parabolic equation that describes the above mentioned gradient flow, we show that this fact is indeed a consequence of the maximum principle.

Based on this we prove two abstract existence results: Theorem 2.1 which concerns the case where G is a finite reflection group and Theorem 2.2 that treats the case of a discrete (infinite) group G .

Theorem 2.1 (Point group, $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$). *Under the hypotheses that*

- W is invariant with respect to a reflection point group Γ , and that the closure of the fundamental domain $\bar{\Phi}$ contains a unique zero of W , say a_1 ,
- there exist: a finite reflection group G acting on \mathbb{R}^n , and a positive homomorphism $f : G \rightarrow \Gamma$ (cf. Definition 1) that associates Φ with the fundamental domain F of G ,

then there exists an f -equivariant solution u of (1.3), $u(gx) = f(g)u(x)$, for $g \in G$, which is positive, and connects the phases at infinity:

$$(2.4) \quad u(\bar{F}) \subset \bar{\Phi} \text{ (positivity),}$$

$$(2.5) \quad |u(x) - a_1| \leq Ce^{-cd(x, \partial D_1)}, \quad x \in D_1,$$

where $D_1 = \text{Int}(\cup_{g \in f^{-1}(\text{Stab}(a_1))} g\bar{F})$.

In the case where a discrete reflection group acts on the domain space, we give a slightly different version of the theorem. Since the fundamental region of a discrete reflection group is bounded or has a cylindrical structure, the exponential estimate applies when the corresponding lattice blows up. By rescaling, this is equivalent to multiplying the gradient of the potential in (1.3) by a factor R^2 , and consider the lattice in the domain space as fixed.

Theorem 2.2 (Lattice). *Under the hypotheses that*

- W is invariant with respect to a reflection point group Γ , and that the closure of the fundamental domain $\bar{\Phi}$ contains a unique zero of W , say a_1 ,
- there exist: a discrete reflection group G acting on \mathbb{R}^n , and a positive homomorphism $f : G \rightarrow \Gamma$ (cf. Definition 1) that associates Φ with the fundamental domain F of G ,

then there exists an R_0 such that for all $R > R_0$, there exists an f -equivariant solution $u_R : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ($u_R(gx) = f(g)u_R(x)$, for $g \in G$) to system

$$(2.6) \quad \Delta u_R - R^2 W_u(u_R) = 0, \text{ for } x \in \mathbb{R}^n,$$

which is positive, and connects the phases:

$$(2.7) \quad u(\overline{F}) \subset \overline{\Phi} \text{ (positivity),}$$

$$(2.8) \quad |u_R(x) - a_1| \leq C e^{-cRd(x, \partial D_1)}, \quad x \in D_1,$$

where $D_1 = \text{Int} \left(\bigcup_{g \in f^{-1}(\text{Stab}(a_1))} g\overline{F} \right)$.

From (2.2) and the f -equivariance of u it follows that

$$(2.9) \quad u(g\overline{F}) \subset f(g)\overline{\Phi}, \quad \text{for } g \in G.$$

For instance, going back to the situation described in Figure 5, let G be the discrete reflection group depicted in Figure 5 left, and let $\Gamma = D_2$ as in Figure 5 right. We consider the positive homomorphism $f : G \rightarrow \Gamma$ such that the reflection along $u_i = 0$ ($i = 1, 2$) in the range, is the image by f of the reflections along $x_i = k$ ($k \in \mathbb{Z}$) in the domain. By taking $F = (0, 1) \times (0, 1)$ (cf. the yellow square in Figure 5 left), and $\Phi = (0, \infty) \times (0, \infty)$ (cf. the yellow quadrant in Figure 5 right), we deduce from (2.2) and the f -equivariance of u , that any square in the domain is mapped into a quadrant with the same colour (cf. Figure 5). Moreover, since we can show that the map u only vanishes on $\mathbb{Z} \times \mathbb{Z}$, we obtain according to Figure 5, the configuration of the vortices of u : at the points $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ where both k and l are either even or odd, u has a vortex of degree $+1$, while at the remaining points $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, u has a vortex of degree -1 . In this way, we can model the experiment presented in Figure 3, where a liquid crystal is illuminated by laser lights centered at the points $2\mathbb{Z} \times 2\mathbb{Z}$. We point out that in this context the vector parameter u is related to the orientation of the molecules. For the general theory of light-matter interaction in liquid crystals we refer to [6].

Therefore, besides its importance for the proofs of Theorem 2.1 and Theorem 2.2, the mapping property (2.2) is a source of information on the geometric structure of the vector valued map u . The fact that (2.9) holds true in general in the abstract setting of the present analysis can perhaps be regarded as one of the significant results in [3]. From the mathematical point of view, symmetry plays an essential role in the derivation of pointwise estimate for solutions of (1.3). Indeed by exploiting the symmetry we show, as in [1] and [2], the existence of minimizers of J_Ω that map *fundamental domains* in the domain x -space into *fundamental domains* in the target u -space. A basic consequence of this is the existence of minimizers that in certain sub-domains avoid all the minima of W but one. It follows that in each such sub-domain the potential W can be considered to have a unique global minimum. This is a key point since, as we have observed, for potentials with two or more global minima, it is very difficult to determine *a priori* in which subregions a minimizer u of J_Ω is near to one or another of the minima of W (cf. [7], [8]).

Due to the variety of choices for n and m (the dimensions of domain and target space), the reflection groups G and Γ , and the homomorphism $f : G \rightarrow \Gamma$, we will deduce from Theorem 2.1 and Theorem 2.2 the existence of various complex multi-phase solutions of (1.3) including several types of lattice solutions. With the help of the pictures below we will briefly describe these constructions detailed in [3].

Our first example is the *triple junction*, a solution of (1.3) that corresponds to the situation where three different phases meet at a junction point. In this case we take $n = m = 2$, $G = \Gamma = D_3$ the symmetry group of the equilateral triangle and $f : D_3 \rightarrow D_3$ the identity. Figure 6 right shows the partition of \mathbb{R}^2 in fundamental domains under the action of D_3 . We assume that the only zero of $W : \mathbb{R}^2 \rightarrow \mathbb{R}$ in $\overline{\Phi}$ is $a_1 = (1, 0)$. Then the invariance of W under D_3 implies that the zeros of W are a_1 , $a_2 = (-1/2, \sqrt{3}/2)$ and $a_3 = (-1/2, -\sqrt{3}/2)$, the vertices of an equilateral triangle with center

O . The stabilizer of a_1 , the subgroup of $\Gamma = D_3$ that leaves a_1 fixed, is $\text{Stab}(a_1) = \{I_2, \sigma_1\}$ with σ_1 the reflection in the line $u_2 = 0$. This and $f = I$ imply that $D_1 = \{x \in \mathbb{R}^2 : x_1 > 0, |x_2| < \sqrt{3}x_1\}$ and Theorem 2.1 yields a solution $\mathbb{R}^2 \ni x \rightarrow u(x) \in \mathbb{R}^2$ of (1.3) that in D_1 converges exponentially to a_1 as the distance $d(x, \partial D_1)$ diverges to $+\infty$. If we interpret Figure 6 right as the partition of the domain space \mathbb{R}^2 under the action of $G = \mathbb{R}^2$, D_1 corresponds to the union of the yellow and the green sectors. The existence of the triple junction solution of (1.3) was first established in [5].

Similarly, for $n = m = 3$, Theorem 2.1 implies the existence of a *quadruple junction*, a solution $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of (1.3) that corresponds to the situation where four different phases meet at a junction point. To see this take $G = \Gamma = \mathcal{T}$, the symmetry group of the tetrahedron A_1, A_2, A_3, A_4 in Figure 6 left and $f = I$. The fundamental domains for the action of \mathcal{T} on \mathbb{R}^3 are the 24 cones with vertex in O generated by the 24 triangles on the boundary of the cube depicted in Figure 6 left. If we assume that $a_1 = A_1$, then the invariance of W under $\Gamma = \mathcal{T}$ implies that W has exactly four zeros that coincide with A_1, A_2, A_3, A_4 . Since $f = I$ we can identify domain and target space and think of the cones in Figure 6 left as the partition of \mathbb{R}^3 both in domain and target space. Then D_1 corresponds to the union of the six cones that have A_1 on their boundary and \mathbb{R}^3 is decomposed in four cones congruent with D_1 , one for each of the A_i and in each of these four cones we have exponential convergence of the solution u to the corresponding A_i . We mention that the existence of the quadruple junction solution of (1.3) was first established in [9].

Actually the abstract character of Theorem 2.1 allows to extend the above results to general $n = m \geq 3$ to show the existence of a $n + 1$ -junction, a solution $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of (1.3) that model the situation where $n + 1$ phases meet at a point. In this general setting $G = \Gamma = \mathcal{T}_n$, with \mathcal{T}_n the symmetry group of the hypertetrahedron with $n + 1$ vertices. W has $n + 1$ zeros that coincide with the vertices of the hypertetrahedron. Now \mathbb{R}^n is divided in $n + 1$ congruent cones one for each of the zeros of W and in each of such cones u converges exponentially to the corresponding zero of W .

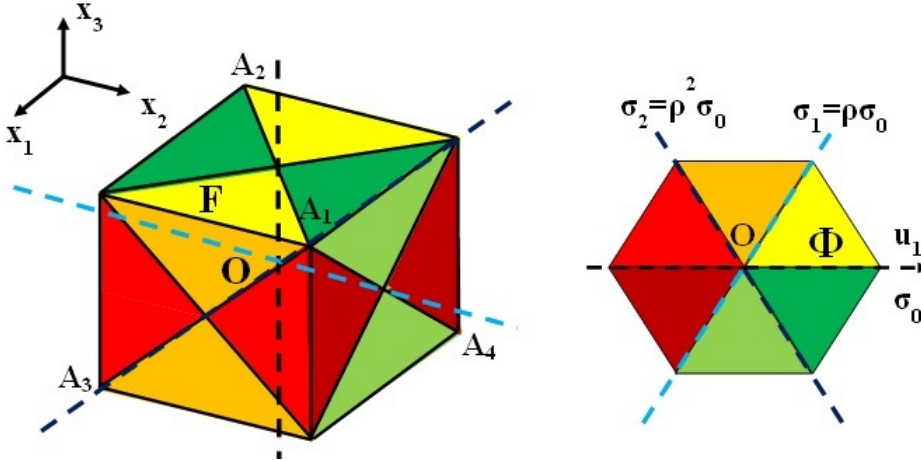


FIGURE 6. Fundamental domains for the action of \mathcal{T} on \mathbb{R}^3 (left) and for the action of D_3 on \mathbb{R}^2 (right). The f -equivariant solution $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of (1.3) given by Theorem 2.1 maps fundamental domains into fundamental domains with the same color. In particular u maps the infinite double cone (union of four fundamental domains) generated by O and by the two yellow triangles into the sector Φ .

Beside the above examples Theorem 2.1 can describe junction points with quite complex structure. We give an example where the dimension of the range ($m = 2$) is intrinsically distinct from the dimension of the domain ($n = 3$). Indeed we take $G = \mathcal{T}$ acting on \mathbb{R}^3 and $\Gamma = D_3$ acting on \mathbb{R}^2 cf. Figure 6.

With s_0 and s_1 the reflections in the plane OA_1A_2 and OA_3A_4 respectively and s_2 the reflection in the plane OA_1A_4 . We consider the positive homomorphism $f : \mathcal{T} \rightarrow D_3$ defined by

$$\begin{aligned} f(s_0) = f(s_1) &= \sigma_0, \quad \text{the reflection in the line } u_1 = 0, \\ f(s_2) &= \sigma_1, \quad \text{the reflection in the line } u_2 = \sqrt{3}u_1. \end{aligned}$$

If W has six zeros, one in the interior of each of the fundamental domains for the action of D_3 on \mathbb{R}^2 , the positive f -equivariant solution $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of (1.3) given by Theorem 2.1 maps fundamental domains into fundamental domains with the same color. In particular the four fundamental domains determined by the two yellow triangle and by their symmetric with respect to O are mapped into $\bar{\Phi}$ and correspond to D_1 . \mathbb{R}^3 is divided in six cones congruent with D_1 and in each of these cones u converges exponentially to the corresponding zero of W . To give an idea of the vector field $\mathbb{R}^3 \ni x \rightarrow u(x) \in \mathbb{R}^2$ identify the plane (u_1, u_2) with the plane through $O \in \mathbb{R}^3$ orthogonal to OA_1 and assume that the axis $u_2 = 0$ is parallel to A_3A_4 . Then

$$\begin{aligned} x \text{ in the plane } OA_3A_4 &\Rightarrow u(x) \text{ parallel to } A_3A_4, \\ x \text{ in the plane } OA_1A_2 &\Rightarrow u(x) \text{ parallel to } A_3A_4, \\ x \text{ in the plane } OA_1A_4 &\Rightarrow u(x) \text{ orthogonal to } OA_1A_4. \end{aligned}$$

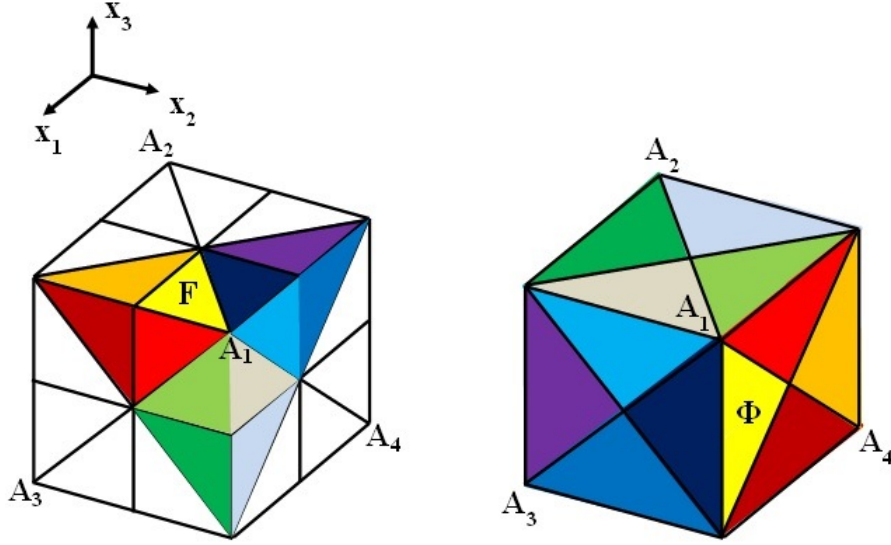


FIGURE 7. Fundamental domains for the action on \mathbb{R}^3 of \mathcal{K} (left) and \mathcal{T} (right). The f -equivariant solution $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of (1.3) given by Theorem 2.1 maps fundamental domains into fundamental domains with the same color.

In the next example, the symmetry group \mathcal{K} of the cube acts on the domain \mathbb{R}^3 , while the symmetry group \mathcal{T} of the tetrahedron acts in the target space \mathbb{R}^3 (cf. Figure 7). We can identify the vertices of the tetrahedron with the vertices A_1, A_2, A_3, A_4 of the cube. Then each symmetry plane of the tetrahedron is also a symmetry plane of the cube. Let s_1 the reflection in the plane OA_1A_2 , s_2 the reflection in the plane OA_1A_4 , s_3 the reflection in the plane $\{x_2 = 0\}$, and s_5 the reflection in the plane OA_2A_3 . Then the homomorphism $f : \mathcal{K} \rightarrow \mathcal{T}$ defined by the conditions:

$$(2.10) \quad f(s_1) = s_1, f(s_2) = s_2, f(s_3) = s_5$$

is a positive homomorphism and any positive f -equivariant map maps fundamental domains into fundamental domains with the same color as indicated in Figure 7.

If W has 24 zeros, one in the interior of each fundamental domain, the set D_1 is the double cone generated by the yellow triangle and the center of the cube in Figure 7 and \mathbb{R}^3 is partitioned in 24 cones congruent with D_1 and in each of these cones the solution u given by Theorem 2.1 converges exponentially to the corresponding zero of W and u is a model for a 24-junction point.

If instead the potential W has for instance only four zeros (located at the vertices of the tetrahedron A_1, A_2, A_3 and A_4), then D_1 is the union of two octants symmetric with respect to the center of the cube and the domain space \mathbb{R}^3 is partitioned in four congruent cones as indicated in Figure 8

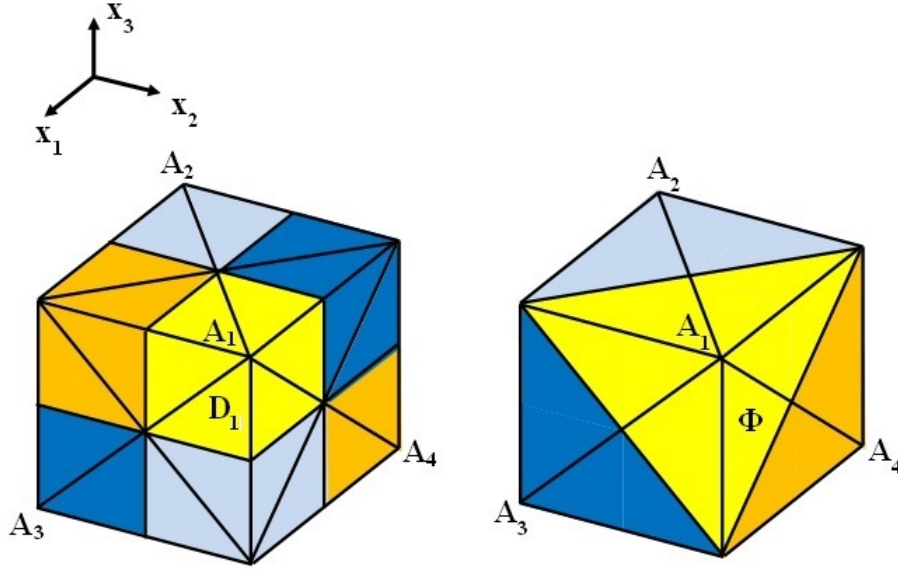


FIGURE 8. The partition of the domain space \mathbb{R}^3 when W has four minima on the vertices of the tetrahedron. In this case $D_1 = \{x \in \mathbb{R}^3 : x_i x_j > 0, 1 \leq i, j \leq 3\}$ and the solution u of (1.3) given by Theorem 2.1 satisfies: $u(x) \rightarrow A_1$, for $\min_i |x_i| \rightarrow +\infty, x \in D_1$.

Next we use Theorem 2.2 to give examples of solutions of (1.3) with a periodic character which model the situation where regions with different phases tile the space and repeat itself with an almost regular pattern. In Figure 5 we have already considered an example of this kind. The

solution $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ considered in Figure 5 is periodic of period 2 in both variables and therefore the pattern in Figure 5 can be generated by applying the translations

$$t_{hk}x = (x_1 + 2h, x_2 + 2k), \quad (h, k) \in \mathbb{Z} \times \mathbb{Z}$$

to the elementary *cell*: the union of the four square that touch the origin O .

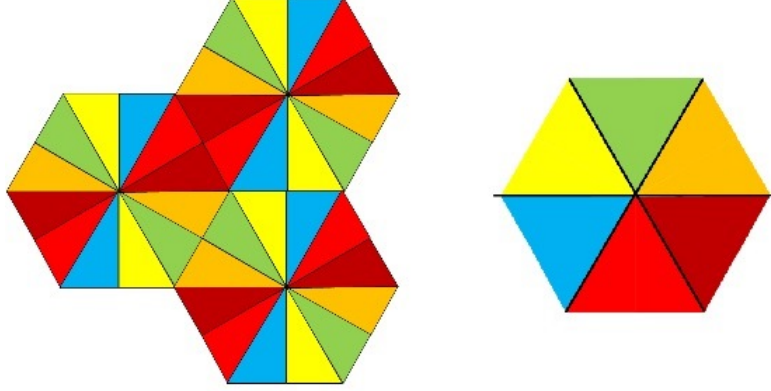


FIGURE 9. Fundamental domains for the actions on \mathbb{R}^2 of G' (left) and D_3 (right). The f -equivariant solution u_R of (1.3) given by Theorem 2.2 maps fundamental domains into fundamental domains with the same color.

In Figure 9 we present a more involved example of lattice solution than the square lattice described in Figure 5 above. In the domain \mathbb{R}^2 we consider the discrete reflection group G generated by: the reflection s_0 in the line $x_2 = 0$, the reflection s_1 in the line $x_2 = x_1/\sqrt{3}$ and the reflection s_2 in the line $x_2 = -\sqrt{3}(x_1 - 1)$. In the target space \mathbb{R}^2 we take $\Gamma = D_3$. In Figure 9 left we show the partition of \mathbb{R}^2 in fundamental domain under the action of G .

The subgroup generated by s_0 and s_1 coincides with D_6 , the symmetry group of the regular hexagon, and with the stabilizer of the origin (the subgroup of the $g \in G$ that leaves O fixed). The discrete reflection group G contains also the translation group T generated by the translations

$$t_{\pm}(x) = x + \left(\frac{3}{2}, \pm \frac{\sqrt{3}}{2}\right).$$

The positive homomorphism $f : G \rightarrow D_3$ is defined by

$$\begin{aligned} f(s_0) &= \text{the reflection in the line } u_2 = 0, \\ f(s_1) &= \text{the reflection in the line } u_2 = -\sqrt{3}u_1, \\ f(s_2) &= \text{the reflection in the line } u_2 = -\sqrt{3}u_1. \end{aligned}$$

Figure 9 describes the mapping properties of an f -equivariant positive solution u_R of (1.3) given by Theorem 2.2. In this case the elementary cell is the hexagon union of the 12 triangles with a vertex at the origin. As hinted in Figure 9, under the action of the translation group T , the elementary cell tiles the whole plane. If we assume that a_1 is in the interior of Φ and therefore that W has exactly six zeros, one in each of the fundamental domains for the action of D_3 on \mathbb{R}^2 , Then the set

D_1 is the union of all yellow triangles in the plane. The solution u depends on $R > 0$ and (2.8) in Theorem 2.2 implies

$$\lim_{R \rightarrow +\infty} u_R(x) = a_1, \quad x \in D_1.$$

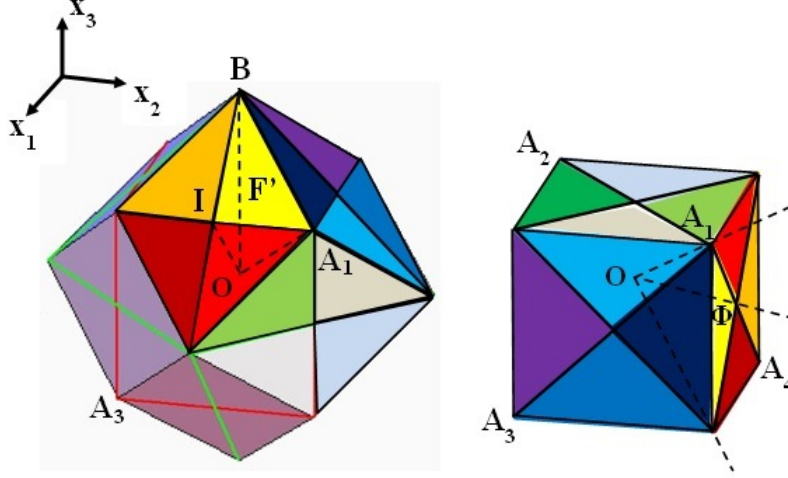


FIGURE 10. Fundamental domains for the action on \mathbb{R}^3 of \mathcal{K}' (left) and \mathcal{T} (right). The fundamental domain F' of \mathcal{K}' is a pyramid with basis the triangle A_1BI and vertex in O . Under the action of the point group \mathcal{K} , F' generates the rhombic dodecahedron C (left) which tiles the domain space \mathbb{R}^3 when translated by the elements of T . The f -equivariant solution $u_R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of (1.3) given by Theorem 2.2 maps fundamental domains into fundamental domains with the same color. Note in particular that u maps $\cup_{t \in T} (F' \cup -F')$ into $\bar{\Phi}$.

Our last example is a solution $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of (1.3) that corresponds to a tiling of the three dimensional space (cf. Figure 10). For this example $n = m = 3$, $\Gamma = \mathcal{T}$ the symmetry group of the tetrahedron and G is the discrete reflection group \mathcal{K}' acting in \mathbb{R}^3 which is generated by the reflections s_1, s_2, s_3 and s_4 in the planes $P_1 := OA_1A_2$, $P_2 := OA_1A_4$, $P_3 := \{x_2 = 0\}$ and $P_4 := \{x_1 + x_3 = 2\}$. These planes bound the fundamental domain F' of \mathcal{K}' with vertices at the points $O, A_1, I := (1, 0, 1)$ and $B := (0, 0, 2)$ (cf. Figure 10). In this case, the positive homomorphism f is defined by (2.10) and by

$$f(s_4) = s_5.$$

The homomorphism f associates F' to the fundamental domain Φ of \mathcal{T} bounded by the planes OA_1A_2, OA_1A_4 and OA_2A_3 . The point group of \mathcal{K}' , that is the stabilizer of the origin, is the group \mathcal{K} , the symmetry group of the cube and we have $\mathcal{K}' = \{tk : k \in \mathcal{K}, t \in T\}$, where T is the translation group of \mathcal{K}' . T is generated by the translations with respect to the vectors $t_1 := (2, 0, 2)$, $t_2 := (0, 2, 2)$ and $t_3 := (0, -2, 2)$. the elementary cell $C = \cup_{g \in \mathcal{K}} g\bar{F}'$ is a *rhombic dodecahedron* (the union of the 48 fundamental domains that touch the origin cf. Figure 10) that tiles the three

dimensional space when translated by the elements of T ¹. Several structures are possible for the solution u_R given by Theorem 2.2 depending on the position of $a_1 \in \bar{\Phi}$. For instance if a_1 is in the interior of Φ , W has 24 zeros and $D_1 = \cup_{t \in T}(F' \cup -F')$ and the space is partitioned in 24 sets congruent with D_1 and, in each of these sets, $u_R(x)$ converges to corresponding zero of W as $R \rightarrow +\infty$.

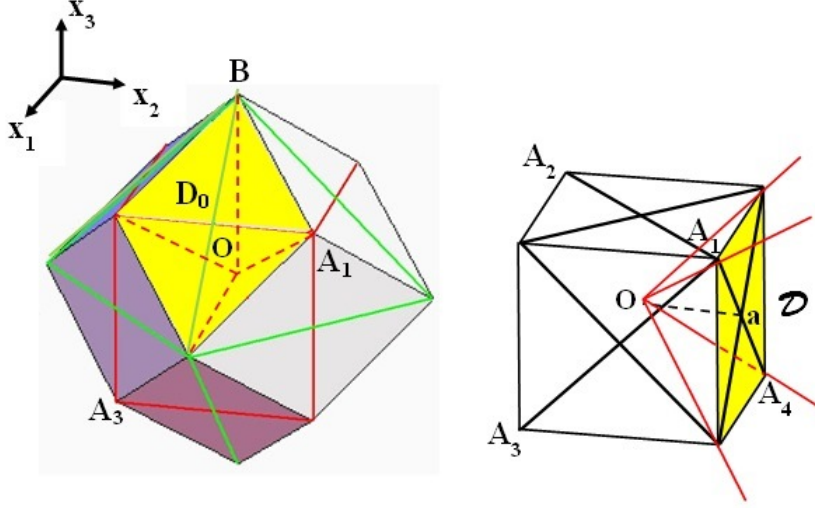


FIGURE 11. If $a_1 = (0, 1, 0)$, W has six minima (one in the middle of each side of the tetrahedron). In this case the positive f -equivariant solution $u_R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by Theorem 2.2 satisfies $\lim_{R \rightarrow +\infty} u_R(x) = (0, 1, 0)$ for $x \in D_1 = \cup_{t \in T}(D_0 \cup -D_0)$, D_0 the pyramid with basis the rhombus defined by the points $A_1, B, (1, -1, 1), (2, 0, 0)$, and vertex in O .

If instead $a = (0, 1, 0)$ W has six zeros the middle points of the sides of the tetrahedron (cf. Figure 11). $D_1 = \cup_{t \in T}(D_0 \cup -D_0)$ with D_0 the pyramid with basis the rhombus defined by the points $A_1, B, (1, -1, 1), (2, 0, 0)$, and vertex in O . Theorem 2.2 implies

$$\lim_{R \rightarrow +\infty} u_R(x) = (0, 1, 0), \quad x \in D_1.$$

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¹Space filling tessellation with rhombic dodecahedra is the crystal structure in which often are found garnets and other minerals like pyrite and magnetite.

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