

# One Dimensional Dynamics

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## Objective

One-dimensional maps are the simplest dynamical systems that may be chaotic. Textbooks can give a static picture of such dynamics, but since dynamics involves time, a student can get a much better understanding of evolution and chaos by using hands on software. There is no substitute for seeing dynamics evolve before your eyes. While there are many general purpose tools that can be programmed to show these dynamics, having a single purpose application allows the instructor and the student to focus on the phenomena, avoiding the programming details.

James Meiss developed and has used the Macintosh application “1DMaps” in both in an undergraduate engineering focused course on dynamics (usually using Steve Strogatz’s wonderful text), and in a graduate course on discrete dynamical systems. Randell Callahan was a student in the undergraduate class last fall, and has been porting the 1DMaps program to iOS so it will run on the iPad, and iPhone platforms.

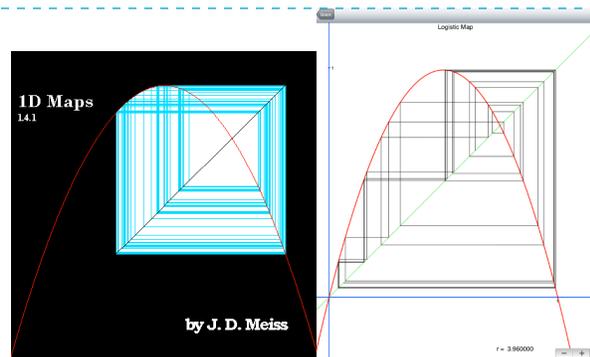
Goals of the lectures, labs and homework assignments that use 1DMaps are (1) to develop an intuition for how one iterates maps (2) visualize the concepts of stability, bifurcation, and asymptotic behavior, and (3) investigate Feigenbaum’s universality.

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## Materials

The Macintosh or iPad application  
“1DMaps”

1. The Mac App is available from <<http://amath.colorado.edu/faculty/jdm/1DMaps>>.
2. The iOS app will soon be available from the Apple iOS app store, initially for the iPad, but soon for the iPhone/iPod as well.



1DMaps for the Macintosh and the iPad

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## Procedures

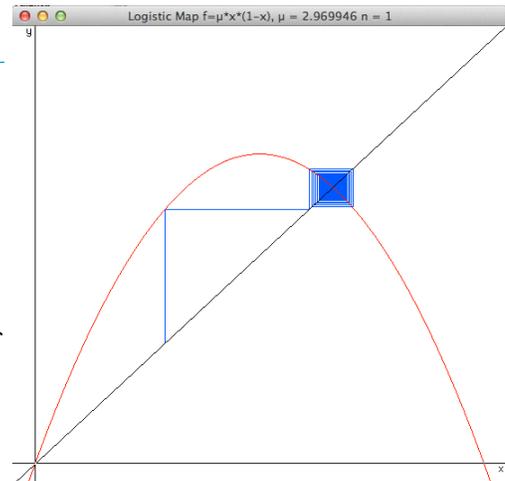
1DMaps is used for in-class demonstrations, a laboratory module, and a homework assignment for a junior-level course on Chaos, typically taught from Strogatz’s book, Nonlinear Dynamics and Chaos. It is also used for as a lecture tool for a graduate class on discrete dynamics.

1. Lecture: Introduction to 1D dynamics and iteration by the cobweb method

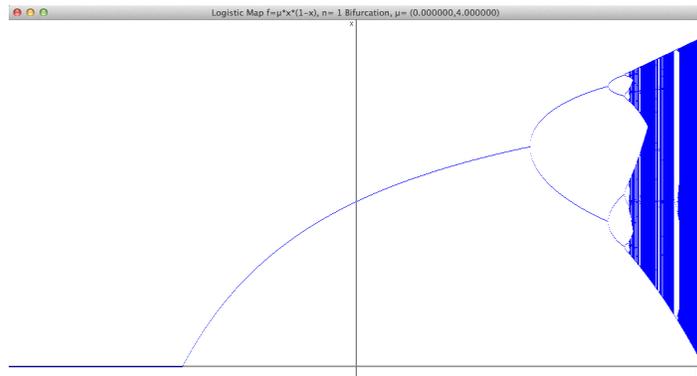
- a. periodicity as fixed points of  $f^n$ ;
- b. stability visualized from the slope and zooming into the local region.
- 2. Lecture: Introduction to Bifurcations
  - a. incrementing parameters leads to bifurcation;
  - b. comparing analytical bifurcation diagram with numerical one: what happens when a fixed point loses stability—period two!
- 3. Lab: Bifurcation and Universality. This lab is run during one class period in a campus computer facility (see attachment, page 4).
- 4. Lecture: Universality
  - a. computing Feigenbaum’s constant;
  - b. comparing various maps, logistic, exponential cosine, etc.
- 5. Homework. Assignments of Strogatz problems in Chapter 10 use maps that are on the menu for 1DMaps.

**Some Visualizations**

- 1. Selecting ⌘-W opens a window to show the cobweb diagram (see figure). Pressing the spacebar iterates the map once. The number of iterates can also be set from the “Parameters” menu (⌘-T). The parameter  $\mu$  can be incremented or decremented using the  $\uparrow$  and  $\downarrow$  keys. In addition, higher iterates of the map can be viewed using the  $\rightarrow$  and  $\leftarrow$  keys.
- 2. Selecting ⌘-B opens a window to show the bifurcation diagram. The map is simply iterated  $t/10$  times for the dynamics to settle down, and then  $9t/10$  iterates are shown. The user can zoom-in (to any plot) by selecting the zoom command (⌘-Z) from the “Window” menu.

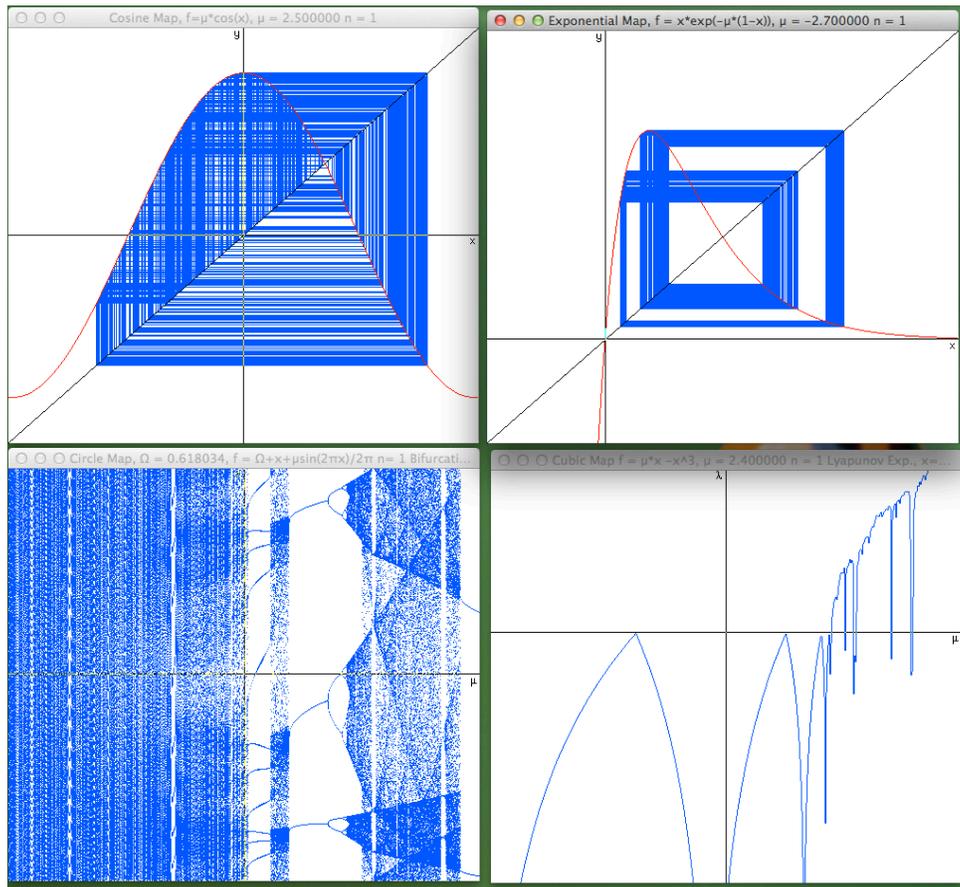


Iteration of the Logistic Map using 1DMaps

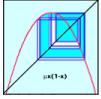


Bifurcation Diagram of the Logistic map with  $t = 50000$ .

3. Additional maps can be shown using the “Choose Map” menu. The figure below shows cobweb diagrams for the (a) cosine-map, and (b) exponential maps. Also a bifurcation diagram for (c) Arnold’s circle map is shown. Finally, the Lyapunov exponent for the cubic map is plotted as a function of the parameter  $\mu$ . Note the negative spikes ( $\mu \rightarrow -\infty$ ) corresponding to superstable periodic orbits.



Iterating multiple maps and multiple views. Cobweb diagrams for (a) the cosine map  $x' = 2 \cos(x)$  and (b) the exponential map  $x' = x \exp(-\mu(1-x))$ . (c) Bifurcation diagram of Arnold's circle map  $x' = x + \Omega + \mu \sin(2\pi x)$ , and (d) Lyapunov exponent as a function of  $\mu$  for the cubic map  $x' = \mu x + x^3$ .



## Lab: One-Dimensional Dynamics

Here we will investigate some of the dynamics for one-dimensional maps using the Mac application *IDMaps*. Though you are encouraged to use this program, it would be possible to do this with other tools, and you certainly may do so if you wish.

### 0) Getting the Program: Begin by downloading the app onto your computer. Go to the Web location

<<http://amath.colorado.edu/faculty/jdm/1DMaps.html>>

Click on the download link for version 1.4.4. This will put a file “1Dmaps1.4.4.zip.” in your Downloads folder (or on your desktop). Double click on it to run the decompression program. Drag the resulting application to the desktop.

You are expected to hand in a report of this lab on Wednesday. The report should contain the answers to the five “tasks” below. Include some snapshots of what you’ve seen in your write-up.

### 1) Beginning the Exploration

- o **Double click on the “1DMaps” file to launch the program.**

You will be presented with two windows, a “plot” window and a “text” window.

You will see the graph of the Logistic map  $f(x) = \mu x(1-x)$  in red and the diagonal  $y = x$  in black. The program uses  $\mu$  instead of  $r$  as the parameter. Also plotted in blue is a typical cobweb diagram. The program sets  $\mu = 3.569946$  initially. We’ll see why below.

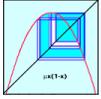
- o **Reset the number of iterates in the orbit to zero using the menu Parameters → Reset Orbit (equivalently ⌘-O).**

Now use the spacebar to iterate the map step by step. You can change the value of  $\mu$  by using the up and down arrow keys on your keyboard (alternatively use the Parameters → Set Parameters menu item...(⌘-K) then you must enter the values of  $\mu$  and  $n$  (use 1 for now) in the parameter dialog.

**Task 1:** Find parameter values at which the stable period  $2^1$ ,  $2^2$ , and  $2^3$  orbits are first created. Label these  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$ .

*Hints:*

- o **Use the “erase transients” command (⌘-E) to discard the first few iterations of your orbit. Then your orbit should eventually settle down to its asymptotic behavior.**
- o **You can refine the parameter increment that the “arrow” keys use by using the “set parameter increment” command (⌘-D).**
- o **Use the command ⌘-Shift-4 to take a snapshot of what is on the screen, dragging the cursor over the desired region. The file, (by default a “png” file) called “Screen Shot...”, will appear on the desktop. It can be inserted into your report (e.g. in MS Word use the Insert → Photo → Picture command, or simply drag the icon of the file to your document). Note: If you hit the spacebar after you press ⌘-Shift-4, then you can take a picture of an entire window upon mouse click.**



- You can resize either window by dragging from the lower right corner. The “resize box” is not drawn in the plot window, but it works nevertheless.

## 2) Bifurcation Diagram

That wasn't the easiest way to find the bifurcation points, and the accuracy of the values is probably not high. It is easier to use the bifurcation diagram. Create this by

- choosing “New Plot” (⌘-N) from the File menu, and then selecting “Bifurcation Diagram” from the Show menu (⌘-B).

The default axes are  $\mu \in [0,4]$  and  $x \in [-0.05,1.05]$  (see the text window). You can zoom in on any graph using the commands in the “Windows” menu. Once you select “Zoom in” (⌘-I) the cursor changes to a magnifying glass. Drag it over the region you would like to see.

**Task 2:** Use this to find the bifurcation ( $\mu$ ) values corresponding to the first few period doublings. Record your values in a table. How high can you go?

You will note that as you do this task you need to iterate the map longer to get an accurate picture.

- Use the “Parameters →Set # Iterates” menu to increase this as you zoom in. Note: if you set it too high—you will see the spinning “wait cursor”—you can cancel the current computation by using ⌘-. (Command period).

**Task 3:** The sequence of bifurcation values appear to be monotonically increasing, but are converging to some finite value of  $\mu$ . Estimate this value.

The limit is called the “period-doubling accumulation point” and was first discovered by Feigenbaum in 1978. We'll talk more about it soon in class. If you are careful, you can compute the limit using a nice trick. Let  $\mu_k$  be the parameter value at which the period  $2^k$  orbit is created. Feigenbaum observed that

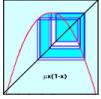
$$\mu_k = \mu_\infty + C\delta^{-k}$$

where  $\mu_\infty$  is the accumulation value and  $\delta$  is the famous *Feigenbaum* constant. We can estimate  $\delta$  by using

$$\delta_i = \frac{\mu_i - \mu_{i-1}}{\mu_{i+1} - \mu_i}$$

Try it! To do this you should make a table that looks something like this (the numbers below are made up):

$i$	$\mu_i$	$\mu_i - \mu_{i-1}$	$\delta_i$
1	2		
2	2.5	0.5	$0.5/0.3=1.66$
3	2.8	0.3	$0.3/0.2=1.50$
4	3.0	0.2	...



What value of  $\delta$  do you get? Once you have estimated  $\delta$ , you can go back to the equation for  $\mu_i$  and estimate  $\mu_\infty$ . What do you get?

### 3) Period 3 and 5

You'll notice that we found period 1, 2, 4, 8, 16, etc. orbits, but never period 3, 5, 7, 9, 10, etc. Where are these? As we will see later in class, *all* of these orbits *must* exist when  $\mu = 4$ . Therefore, they must be created somewhere. Here we'll find period 3 and 5.

**Task 4:** Use the bifurcation diagram to search for the creation of period 3 and period 5 orbits. Blow up the bifurcation diagram over the region that these orbits exist. What else is going on here?

You'll have to look in the chaotic region beyond the period-doubling accumulation,  $\mu_\infty$ . Notice that this region is not uniformly filled—there are some empty regions. Inside these apparently empty regions, there is an incredibly complex bifurcation pattern!

Now go back to the cobweb picture. Select the “Set Parameters” item and type in a value of  $\mu$  near the creation of the period-3 orbit. Also set  $n = 3$ . This means that the picture will draw  $f^3$  instead of  $f$ . (You can also use the right and left arrow keys to increment and decrement  $n$ ).

Finally set the “parameter increment” to a small value, so you can step through the region where there is a stable period three orbit.

**Task 5:** Show that  $f^3$  crosses the diagonal in three new places just where the period three orbit is created. How many period three orbits are there? Which of these is stable and which is unstable? Increase the parameter value until the period three orbits are all unstable. What happens there?